

Polarization Elements—A Group Theoretical Study

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Abstract

The classification of polarization elements, the polarization affecting optical devices which have a Jones-matrix representation, according to the type of eigenvectors they possess, is given a new visit through the Group-Theoretical connection of polarization elements. The diattenuators and retarders are recognized as the elements corresponding to boosts and rotations, respectively. The structure of homogeneous elements other than diattenuators and retarders are identified by giving the quaternion corresponding to these elements. The set of degenerate polarization elements is identified with the so called ‘null’ elements of the Lorentz Group. Singular polarization elements are examined in their more illustrative Mueller matrix representation and finally the eigenstructure of a special class of singular Mueller matrices is studied.

Key words: *Homogeneous, Inhomogeneous, Degenerate polarization elements, Eigenpolarization, Lorentz Group, Quaternions, Singular Mueller matrices*

1 Introduction

It is well known that [1, 2] polarization elements are characterized by the types of eigenpolarization that they possess. *Homogeneous polarization elements* are the ones which possess orthogonal eigenpolarization whereas *inhomogeneous polarization elements* possess non-orthogonal eigenpolarizations. Here we refer to a *polarization element* as a polarizing optical device which has got the Jones matrix or the 2×2 matrix representation. Also, the term eigenpolarization refers to the eigenvectors of the associated Jones matrix, the states which are unchanged in polarization by the action of the corresponding Jones device matrix. There is another class of polarization elements that are called *degenerate* polarization elements. They are the ones which possess only one linearly independent eigenpolarization.

Though the connection between the Lorentz group and polarization elements is well known [3, 4, 5, 6], little has been done in exploiting the known properties of the Lorentz group in identifying the homogeneous, inhomogeneous and degenerate elements. Ours is an attempt towards this end and by achieving this, we hope to have lessened the jargon to those who are familiar with Group Theory.

We recall here the connection between the set of all non-singular polarization elements and the Lorentz group [4].

A non-singular pure Mueller matrix is of the form kL where L is an element belonging to the Orthochronous Proper Lorentz Group (OPLG) $SO(3,1)$, with k being any positive real number.

Though the connection is always stated in terms of 4×4 representation of the polarizing optical devices, the so called Mueller matrix representation, and the group $SO(3,1)$, the connection between the 2×2 matrix representation of the device (Jones representation) and the group $SL(2,C)$ is obvious through the homomorphism between the groups $SL(2,C)$ and $SO(3,1)$. In fact, there is a one-to-many mapping between the set of all $SL(2,C)$ matrices and the set of all Jones matrices, a Jones matrix being just a complex scalar times a $SL(2,C)$ matrix. This being the case, a classification of $SL(2,C)$ into homogeneous and inhomogeneous elements gives us the corresponding classification of non-singular Jones matrices.

2 Classification of elements of $SL(2,C)$ on the basis of their eigenvectors

We know that among the elements of $SL(2,C)$, we have those elements which are unitary and hence correspond to the subgroup $SU(2)$ of $SL(2,C)$. The elements of $SU(2)$ written in their quaternionic [7] representation¹ are given by

$$\mathbf{T}_r = \begin{pmatrix} q_0 - iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & q_0 + iq_3 \end{pmatrix} \quad (2.1)$$

where q_0, q_1, q_2 and q_3 are all real and $\mathbf{q} \cdot \mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. It is very easy to notice that \mathbf{T}_r has got two eigenvalues $\lambda = q_0 \pm \sqrt{q_0^2 - 1}$ and owing to the relations [7] $q_0 = \cos \frac{\theta}{2}$, $q_i = a_i \sin \frac{\theta}{2}$ ($i = 1, 2, 3$), $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = 1$, where $\hat{\mathbf{a}} = \{a_1, a_2, a_3\}$ is a real (unit) vector, we have $\lambda_{1,2} = \exp \pm \frac{i\theta}{2}$. The eigenvectors of \mathbf{T}_r belonging to these eigenvalues are found to be

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ \frac{a_1 + ia_2}{a_3 - 1} \end{pmatrix}; \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ \frac{a_1 + ia_2}{a_3 + 1} \end{pmatrix} \quad (2.2)$$

which satisfy $\mathbf{X}_1^\dagger \mathbf{X}_2 = \mathbf{0}$. Thus it is clear that all 2×2 unitary matrices, which are a complex scalar times that of the elements of the group $SU(2)$, are homogeneous polarization elements. These are the so called *rotators* which is obviously so because of the known homomorphism between the group $SU(2)$ and the subgroup $1 \oplus \mathbf{R}_3$ of the group $SO(3,1)$ with $\mathbf{R}_3 \in SO(3)$.

Similarly we consider another important set of elements belonging to $SL(2,C)$. These are the 2×2 matrices (of unit determinant) which are hermitian and are represented in terms of a (unit) quaternion $\mathbf{q} = (q_0, \vec{\mathbf{q}})$ as

$$\mathbf{T}_b = \begin{pmatrix} q_0 - iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & q_0 + iq_3 \end{pmatrix} \quad (2.3)$$

with q_0 real and q_1, q_2, q_3 being purely imaginary. The eigenvectors of this matrix are found to be $\lambda = q_0 \pm \sqrt{q_0^2 - 1}$ and owing to the relations [7] $q_0 = \cosh \frac{\theta_b}{2}$, $q_i = n_i \sinh \frac{\theta_b}{2}$, $i = 1, 2, 3$ we have $\lambda_{1,2} = \exp \pm \frac{\theta_b}{2}$. The eigenvalues of this matrix are given by

$$\mathbf{X}'_1 = \begin{pmatrix} 1 \\ \frac{n_1 + in_2}{n_3 + 1} \end{pmatrix}; \quad \mathbf{X}'_2 = \begin{pmatrix} 1 \\ \frac{n_1 + in_2}{n_3 - 1} \end{pmatrix} \quad (2.4)$$

where $\hat{\mathbf{n}} = \{n_1, n_2, n_3\}$ is a real unit vector and $\mathbf{X}'_1, \mathbf{X}'_2$ are mutually orthogonal. We thus have no hesitation in concluding that all hermitian 2×2 matrices are homogeneous. The angle θ_b is called the boost angle because of the homomorphism that exists between hermitian elements of $SL(2,C)$ and the set of all boost matrices belonging to the group $SO(3,1)$. The elements of the group $SO(3,1)$ corresponding to the hermitian elements of the group $SL(2,C)$ being given by

$$\mathbf{M}_b = \mathbf{A}(\mathbf{T}_b \otimes \mathbf{T}_b^*)\mathbf{A}^{-1}; \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix}, \quad (2.5)$$

¹The quaternions are mathematical objects of the form $\mathbf{q} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3 + q_0 \equiv \vec{\mathbf{q}}\mathbf{e} + q_0$ where $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 are symbols called the *quaternion units* satisfying the relations $\mathbf{e}_i^2 = -1$, ($i = 1, 2, 3$), $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_k$ ($i \neq j \neq k$ and i, j, k cyclic). The quaternions over the field of complex numbers are called complex quaternions and they form a group under multiplication. A detailed discussion on quaternions and how they form a representation of the Lorentz group can be found in [7]

the Stokes vectors corresponding to the eigenvectors \mathbf{X}'_1 and \mathbf{X}'_2 are respectively given by

$$\mathbf{S}'_1 = \mathbf{A}(\mathbf{X}'_1 \otimes \mathbf{X}'_1^*) = \{1, n_3, n_1, n_2\}, \quad \mathbf{S}'_2 = \mathbf{A}(\mathbf{X}'_2 \otimes \mathbf{X}'_2^*) = \{1, -n_3, -n_1, -n_2\}. \quad (2.6)$$

\mathbf{S}'_1 and \mathbf{S}'_2 being the eigenvectors of \mathbf{M}_b corresponding to the eigenvalues $\exp \theta_b$ and $\exp(-\theta_b)$ respectively, the reason why these matrices are called diattenuators is obvious. They transmit the orthogonal vectors (orthogonal in the usual sense, not in the Minkowski sense) \mathbf{S}'_1 and \mathbf{S}'_2 with different amounts of absorption ($\exp \theta_b$ for \mathbf{S}'_1 and $\exp(-\theta_b)$ for \mathbf{S}'_2). For the sake of completeness, we write down the Stokes vectors \mathbf{S}_1 and \mathbf{S}_2 corresponding to the orthogonal eigenpolarizations $\mathbf{X}_1, \mathbf{X}_2$ of \mathbf{T}_r . They are,

$$\mathbf{S}_1 = \{1, -a_3, -a_1, -a_2\} \quad \text{and} \quad \mathbf{S}_2 = \{1, a_3, a_1, a_2\}, \quad (2.7)$$

and it is to be noted that whereas $\mathbf{X}_1 \in \exp \theta/2$ and $\mathbf{X}_2 \in \exp(-\theta/2)$, the Stokes vectors \mathbf{S}_1 and \mathbf{S}_2 belong to the eigenvalues (doubly repeated) 1, 1 of \mathbf{M}_r . The remaining two eigenvectors \mathbf{S}_3 and \mathbf{S}_4 of \mathbf{M}_r belonging to the eigenvalues $\exp i\theta, \exp(-i\theta)$ can be seen to be complex 4-vectors thus not corresponding to physical light beams. Similarly the eigenvectors \mathbf{S}'_3 and \mathbf{S}'_4 belonging to the doubly repeated eigenvalues 1, 1 of the boost matrix do not correspond to physical light beams as both of them are not Minkowskian vectors.

Having thus arrived at the conclusions that rotation and boost matrices are the homogeneous polarization elements having orthogonal eigenvectors, we now wish to see which other elements of the Lorentz group correspond to homogeneous polarization elements. To make this examination, we find it useful to recall the well-known polar decomposition theorem [8] realised in the case of the group $\text{SL}(2, \mathbb{C})$. We notice that any element \mathbf{T} of the group $\text{SL}(2, \mathbb{C})$ can be written as a product of a boost matrix \mathbf{T}_b (or \mathbf{T}'_b) and a rotation matrix \mathbf{T}_r as shown below.

$$\mathbf{T} = \mathbf{T}_r \mathbf{T}_b = \mathbf{T}'_b \mathbf{T}_r \quad (2.8)$$

The eigenvectors belonging to \mathbf{T}_r and \mathbf{T}_b respectively being given by equation (2.2) and (2.4), it is not difficult to see that for \mathbf{T} to possess orthogonal eigenvectors, one should have $\hat{\mathbf{a}} = \hat{\mathbf{n}}$. Also, by using the quaternionic representation of the Lorentz group, one can very easily get at the general form of the polarization element possessing orthogonal eigenpolarizations. Since the quaternions corresponding to $\mathbf{T}_r, \mathbf{T}_b$ are, respectively, $\mathbf{q}_r = (\cos \frac{\theta_r}{2}, \hat{\mathbf{a}} \sin \frac{\theta_r}{2})$ and $\mathbf{q}_b = (\cosh \frac{\theta_b}{2}, i\hat{\mathbf{n}} \sinh \frac{\theta_b}{2})$, by using the rule of multiplication of quaternions and using the condition $\hat{\mathbf{a}} = \hat{\mathbf{n}}$, we get at the quaternion \mathbf{q} corresponding to \mathbf{T} , which is homogeneous. It is given by

$$\begin{aligned} \mathbf{q} &= \mathbf{q}_r \mathbf{q}_b; \quad \mathbf{q} = (q_0, \vec{\mathbf{q}}); \\ q_0 &= \cos \frac{\theta_r}{2} \cosh \frac{\theta_b}{2} - i \sin \frac{\theta_r}{2} \sinh \frac{\theta_b}{2}, \\ \vec{\mathbf{q}} &= \left(\cos \frac{\theta_r}{2} \sinh \frac{\theta_b}{2} + \sin \frac{\theta_r}{2} \cosh \frac{\theta_b}{2} \right) \hat{\mathbf{a}}. \end{aligned} \quad (2.9)$$

One can very easily see that when $\theta_r = 0$, \mathbf{q} given above reduces to \mathbf{q}_b , the quaternion corresponding to a boost and when $\theta_b = 0$, \mathbf{q} reduces to \mathbf{q}_r , the quaternion corresponding to a rotation. When both θ_r and θ_b are non-zero, the quaternion which obeys (2.9) is the one corresponding to the general homogeneous polarization element. At this stage, an observation regarding the classification of the group $\text{SO}(3,1)$ depending on the geometric structure its elements possess, may be in order. We recall that [7] if an element $\mathbf{L} = \exp(\mathbf{S})$ of $\text{SO}(3,1)$, where \mathbf{S} is the so-called *infinitesimal transformation matrix* of \mathbf{L} , has an additional structure with

$$\mathbf{S} \equiv \mathbf{Y}\tilde{\mathbf{X}} - \mathbf{X}\tilde{\mathbf{Y}}; \quad \tilde{\mathbf{X}}\mathbf{Y} = 0,$$

\mathbf{X}, \mathbf{Y} being Minkowski 4-vectors, then it is called a *planar* Lorentz transformation. All other elements of $\text{SO}(3,1)$ which do not have their corresponding \mathbf{S} in the form given above are called *non-planar* Lorentz transformations. A corresponding classification of the group $\text{SL}(2, \mathbb{C})$, though this classification is significant mostly in the 4×4 representation of the Lorentz group, is obvious. Also, depending on the 4-vector character of the vectors \mathbf{X} and \mathbf{Y} , we have the so-called [7] *rotation-like*, *null* and *boost-like* planar transformations. The quaternions corresponding to planar and non-planar Lorentz transformations have been identified [7] and it is seen that 'rotations' and 'boosts' which are identified

to be homogeneous polarization elements are respectively special cases of ‘rotation-like’ and ‘boost-like’ Lorentz transformations [7]. But it is interesting to note that **none of the other planar transformations are homogeneous**. This can be seen by observing that the quaternions corresponding to planar Lorentz transformations have q_0 real as the only general condition on them, with their other quaternion components q_1 , q_2 and q_3 being permitted to take any values subject to the condition $\mathbf{q} \cdot \mathbf{q} = 1$. A careful observation of equation (2.9) reveals that the quaternions corresponding to planar Lorentz transformations do not coincide with the form of the quaternion given in that equation (equation (2.9)) for any value of θ_r and θ_b except when either $\theta_r = 0$ or $\theta_b = 0$. The situations in equation (2.9) when $\theta_b = 0$ corresponding to the quaternion representing a rotation and the other situation when $\theta_r = 0$ corresponding to the quaternion representing a boost, our assertion made above is proved. Thus, with the set of all homogeneous elements in the Lorentz group being identified by equation (2.9), it containing planar Lorentz transformations only in the form of rotations and boosts, the remaining elements corresponding to non-planar Lorentz transformations, we can conclude that planar Lorentz transformations with the exception of rotations and boosts are either inhomogeneous or degenerate polarization elements. In the following we try to identify the set of all degenerate polarization elements in the Lorentz group.

2.1 Non-singular Degenerate Polarization elements

We start with checking the so called *null* elements [7] of the Lorentz group for their ‘*degenerate*’ness. The null elements of the Lorentz group are the ones which have their corresponding quaternion as

$$\mathbf{q}_n = (q_0, \bar{\mathbf{q}}); \quad q_1^2 + q_2^2 + q_3^2 + q_0^2 = 1, \quad q_1^2 + q_2^2 + q_3^2 = 0. \quad (2.10)$$

One can easily check that for all elements of the Lorentz group having the quaternion of the above form, the number 1 appears as the doubly degenerate eigenvalue. The eigenvector belonging to this doubly repeated eigenvalue can be seen to be

$$\mathbf{X}_0 = \begin{pmatrix} 1 \\ \frac{-iq_1}{iq_2 + q_3} \end{pmatrix}, \quad (2.11)$$

thus taking the null elements \mathbf{T}_n into the class of degenerate polarization elements. None of the other elements of the Lorentz group are degenerate as evidenced by the fact that only the null elements of the group have doubly degenerate eigenvalues. This result and the discussions made hitherto make it clear that the ‘null’ elements of the Lorentz group are the only non-singular degenerate polarization elements (apart from a scale factor). Having thus identified the *disjoint* sets of the Lorentz group corresponding to homogeneous and degenerate polarization elements, we can now conclude that all other elements of the Lorentz group belong to the class of inhomogeneous polarization elements. Or to put it more simply, the elements of the Lorentz group which have the corresponding quaternions other than the ones mentioned in equations (2.9) and (2.10), correspond to inhomogeneous polarization elements.

It is worthwhile to point out here that all the examples of inhomogeneous and degenerate polarization elements that are quoted in [2] are singular elements though there is a whole lot of non-singular inhomogeneous as well as degenerate polarization elements as we have pointed out here. In Table 1 we give a few illustrative examples of homogeneous, inhomogeneous and degenerate polarization elements which are non-singular.

3 Singular Mueller matrices

Having till now studied the eigenstructure and hence the classification of non-singular polarization elements, we now seek to determine the eigenstructure of singular polarization elements. It is to be noted here that a class of the widely studied and used polarizing optical devices, the so called *polarizers and analyzers*² are singular. Thus it is worthwhile to study their eigenstructure and eventually classify

²For a discussion on polarizers and analyzers, see [9, 10].

them on that basis. Here we wish to work in the 4×4 representation of the devices only as it is convenient mathematically for the task we have on hand.

We recall that *singular pure Mueller matrices* have the form

$$\begin{aligned} \mathbf{M} &= m_{00} \mathbf{X} \tilde{\mathbf{Y}} \mathbf{G}, \quad \text{where } \mathbf{X} = \{1, x, y, z\}; \tilde{\mathbf{X}} \mathbf{G} \mathbf{X} = 0, \\ \mathbf{Y} &= \{1, -p, -q, -r\}; \tilde{\mathbf{Y}} \mathbf{G} \mathbf{Y} = 0, \text{ and } \mathbf{G} = \text{diag}(1, -1, -1, -1). \end{aligned} \quad (3.1)$$

It can be seen that matrices of the form $\mathbf{M} = m_{00} \mathbf{X} \tilde{\mathbf{Y}} \mathbf{G}$ possess the following eigenvalues

$$\lambda_1 = m_{00}(1 + px + qy + rz), \lambda_2 = 0, 0, 0. \quad (3.2)$$

Thus, there are only two distinct eigenvalues for \mathbf{M} of the form (3.1) in general and a quadruply repeated eigenvalue $\lambda = 0$ in the case where $\mathbf{X} = \mathbf{Y}$. Correspondingly, in general, there are two eigenvectors $\mathbf{S}_1 = \mathbf{X} \in \lambda_1$ and $\mathbf{S}_2 = \mathbf{Y} \in \lambda_2$, the case of there being only one eigenvector $\mathbf{Y} \in \lambda = 0$ getting realized when $\mathbf{X} = \mathbf{Y}$. Since $\tilde{\mathbf{S}}_1 \mathbf{S}_2 = \tilde{\mathbf{X}} \mathbf{Y}$ is the analogue of the expression $\mathbf{X}_1^\dagger \mathbf{X}_2$, where \mathbf{X}_1 and \mathbf{X}_2 are the eigenvectors of the Jones matrix corresponding to \mathbf{M} in equation (3.1), singular pure Mueller matrices fall into the class of homogeneous, inhomogeneous or degerate polarization elements depending on the vectors $\mathbf{X} = \mathbf{S}_1$ and $\mathbf{Y} = \mathbf{S}_2$ on which they are built. They are homogeneous when $\tilde{\mathbf{X}} \mathbf{Y} = 0$, inhomogeneous when $\tilde{\mathbf{X}} \mathbf{Y} \neq 0$ and degenerate when $\mathbf{X} = \mathbf{Y}$. In Table 2, we give few examples of homogeneous, inhomogeneous and degenerate polarization elements which are singular. One can also see [2] for several examples of singular inhomogeneous polarization elements.

Though we do not have corresponding Jones matrices for singular Mueller matrices other than the ones mentioned in equation (3.1), we find it worthwhile to examine the eigenstructure of a special class \mathcal{M} of singular Mueller matrices which have the same structure as that of the singular pure Mueller matrices, but the 4-vector character of the composite vectors \mathbf{X} and \mathbf{Y} are different from that of singular pure Mueller matrices. We have three cases to consider depending on the choices possible for the 4-vectors \mathbf{X} and \mathbf{Y} .

- (i) Here $\mathbf{M}_1 \in \mathcal{M}$ is of the form

$$\begin{aligned} \mathbf{M}_1 &= m_{00} \mathbf{X} \tilde{\mathbf{Y}} \mathbf{G}, \quad \text{where } \tilde{\mathbf{X}} \mathbf{G} \mathbf{X} = 0, \\ \tilde{\mathbf{Y}} \mathbf{G} \mathbf{Y} &> 0, \text{ and } \mathbf{G} = \text{diag}(1, -1, -1, -1). \end{aligned} \quad (3.3)$$

It is easy to see that this matrix has only one non-zero eigenvalue $\lambda_1 = m_{00}(1 + px + qy + rz)$ its corresponding eigenvector being $\mathbf{S}_1 = \mathbf{X}$. The eigenvectors corresponding to its triply repeated zero eigenvalue $\lambda_2 = 0$ can be seen to be non-Stokes. It may be of some interest to note that this matrix is the so called *generalized polarizer matrix* [9].

- (ii) Consider a 4×4 matrix \mathbf{M}_2 where

$$\begin{aligned} \mathbf{M}_2 &= m_{00} \mathbf{X} \tilde{\mathbf{Y}} \mathbf{G}; \quad \tilde{\mathbf{X}} \mathbf{G} \mathbf{X} > 0, \\ \tilde{\mathbf{Y}} \mathbf{G} \mathbf{Y} &= 0, \text{ and } \mathbf{G} = \text{diag}(1, -1, -1, -1). \end{aligned} \quad (3.4)$$

This matrix also possesses only one non-zero eigenvalue It is easy to notice that \mathbf{M}_2 has a structure equivalent to that of the transpose of \mathbf{M}_1 . It has got two eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 = 0$ and the corresponding eigenvectors are $\mathbf{S}_1 = \mathbf{X} \in \lambda_1$, $\mathbf{S}_2 = \mathbf{Y} \in \lambda_2 = 0$. The form of the above matrix itself suggests that it belongs to the so called *generalized analyzer matrix* [9].

- (iii) The only other remaining possibility in the choice of \mathbf{X} and \mathbf{Y} being $\tilde{\mathbf{X}} \mathbf{G} \mathbf{X} > 0$ and $\tilde{\mathbf{Y}} \mathbf{G} \mathbf{Y} > 0$, we have

$$\begin{aligned} \mathbf{M}_3 &= m_{00} \mathbf{X} \tilde{\mathbf{Y}} \mathbf{G}; \quad \tilde{\mathbf{X}} \mathbf{G} \mathbf{X} > 0, \\ \tilde{\mathbf{Y}} \mathbf{G} \mathbf{Y} &> 0, \text{ and } \mathbf{G} = \text{diag}(1, -1, -1, -1). \end{aligned} \quad (3.5)$$

On the same lines of that of the previous two cases, we can see that the matrix \mathbf{M}_3 has two distinct eigenvalues, a non-zero eigenvalue λ_1 and a zero eigenvalue $\lambda_2 = 0$. The only eigenvector corresponding to λ_1 is \mathbf{X} whereas the eigenvectors belonging to the triply repeated eigenvalue $\lambda_2 = 0$ are all seen to be non-Stokes vectors and thus not qualifying to be called eigenpolarizations.

We wish to make an observation here on the importance of studying the eigenstructure of Mueller matrices. The eigenvector which correspond to a real eigenvalue of a given Mueller matrix represents the physical light beam that comes out undisturbed by the polarizing optical device represented by the Mueller matrix and hence a study of the eigenstructure of Mueller matrices, at least in the cases possible, is welcome for understanding the nature of the optical devices represented by them. In fact, the singular Mueller matrices that we have studied here is one class of Mueller matrices whose eigenstructure can be studied quite easily. But there still remain a whole lot of Mueller matrices, singular as well as non-singular, which remain to be examined for their eigenstructure. Among the class of non-singular Mueller matrices, we have carried out a study of the pure Mueller matrices which are elements of the Lorentz group (apart from a scale factor) and are in pursuit of other non-singular Mueller matrices which are accessible for a study of their eigenstructure.

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Table 1: Non-singular Polarization Elements

Type of polarization element	Example 1	Example 2
Homogeneous	$\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} - i & 1 - \sqrt{2}i \\ 1 - \sqrt{2}i & \sqrt{2} - i \end{pmatrix}$ $\mathbf{X}_1 = \frac{1}{\sqrt{2}}\{1, -1\}; \mathbf{X}_2 = \frac{1}{\sqrt{2}}\{1, 1\}$	$\frac{1}{2} \begin{pmatrix} 2 - 3i & -2\sqrt{3} - \sqrt{3}i \\ 2\sqrt{3} + \sqrt{3}i & 2 - 3i \end{pmatrix}$ $\mathbf{X}_1 = \frac{1}{\sqrt{2}}\{1, -i\}; \mathbf{X}_2 = \frac{1}{\sqrt{2}}\{1, i\}$
Inhomogeneous	$\begin{pmatrix} -i & -1 - 2i \\ 1 & 2 + i \end{pmatrix}$ $\mathbf{X}_1 = \frac{1}{\sqrt{2}}\{1, -1\}; \mathbf{X}_2 = \frac{\sqrt{5}}{\sqrt{6}}\{1, \frac{-1}{1+2i}\}$	$\begin{pmatrix} 2 + \sqrt{3} & 0 \\ 2(1 - i) & 2 - \sqrt{3} \end{pmatrix}$ $\mathbf{X}_1 = \frac{\sqrt{3}}{\sqrt{5}}\{1, \frac{1-i}{\sqrt{3}}\}; \mathbf{X}_2 = \{0, 1\}$
Degenerate	$\begin{pmatrix} -i & 1 - i \\ 1 - i & 2 + i \end{pmatrix}$ $\mathbf{X}_1 = \mathbf{X}_2 = \frac{1}{\sqrt{2}}\{1, i\};$	$\begin{pmatrix} 1 & 0 \\ 2(1 - i) & 1 \end{pmatrix}$ $\mathbf{X}_1 = \mathbf{X}_2 = \{0, 1\};$

Table 2: Singular Polarization Elements

Type of polarization element	Example 1	Example 2
Homogeneous	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\mathbf{X}_1 = \frac{1}{\sqrt{2}}\{1, 1\}; \mathbf{X}_2 = \frac{1}{\sqrt{2}}\{1, -1\}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{X}_1 = \{0, 1\}; \mathbf{X}_2 = \{1, 0\}$
Inhomogeneous	$\begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}$ $\mathbf{X}_1 = \frac{1}{\sqrt{2}}\{1, 1\}; \mathbf{X}_2 = \frac{1}{\sqrt{2}}\{1, i\}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\mathbf{X}_1 = \frac{1}{\sqrt{2}}\{1, 1\}; \mathbf{X}_2 = \{0, 1\}$
Degenerate	$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ $\mathbf{X}_1 = \mathbf{X}_2 = \frac{1}{\sqrt{2}}\{1, 1\};$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\mathbf{X}_1 = \mathbf{X}_2 = \{1, 0\};$